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THE HOMOGENEOUS VECTOR FUNCTION

AND

DETERMINANTS OF THE P-TH CLASS

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CONTENTS

I. THE HOMOGENEOUS VECTOR FUNCTION

1. *General Outline*

The object of the present paper is to present the properties of a vector function, which may be considered as the generalization of the linear vector function. At the same time we shall attempt to develop the various analytical representations, and to a certain extent indicate the applications.

Our homogeneous vector function of the mth degree is defined as satisfying the relation:

$$k^m \cdot \varphi(\boldsymbol{r}) = \varphi(k\boldsymbol{r}),$$

where $\boldsymbol{r}$ is any vector, and k is a constant.

Rather more general would be a vector function satisfying the relation:

$$f(k) \cdot \varphi(\boldsymbol{r}) = \varphi(k\boldsymbol{r}).$$

We shall not in this part specifically occupy ourselves with the latter, beyond indicating that it may in general be expressed in terms of the former. Moreover we may, and shall, confine ourselves to the case when m is a positive integer. The case when m is a negative integer involves the investigation of the inverse function, and will be treated under that head.

In this present part we further confine ourselves to a method of representation analogous to that of matrices or dyadics, employed in the case of the linear vector function. It will appear that there is a marked contrast between the properties of the latter and those of the general vector function. However in this case also there present themselves singular vectors which play much the same rôle as the axes of the linear vector function, and are in consequence assigned the same term. On the other hand those characteristic numbers which are termed the latent roots of the linear vector function, will readily be seen to have no analogues in the general case. We shall by this method, partly for their own sake, partly for the sake of later comparison, when a different mode of representation, that by means of quarternions, is studied, develop certain of the most fundamental properties, which we believe to have an interest of their own.

In the second part will be given the algebraic theory leading to the determination of the axes of these vector functions. The method will be found closely analogous to that employed in the case of the algebraic theory of eliminants, and its results are, we believe, of some value in the theory of differential equations.

The third part is to be confined to a survey of the geometrical applications, the vector function being regarded as determining a transformation of space.

The fourth part is to be devoted to a brief development of the theory by way of quaternions, and to a comparison of results.

2. *Survey of Applications*

The concept of the vector function enters into three branches of mathematics. In mathematical physics are encountered two distinct types. The first is that determined by two localized vector sheafs, functionally related. As examples of this type may be quoted the relation between the stress and strain vectors in the theory of elasticity, and which, it may be stated in passing, is completely and adequately represented by the self-conjugate linear vector function. Further, there present themselves such relations as that existing between the vectors of electric force and displacement, or between magnetic force and induction. In

these latter cases a more general form of representation is possibly called for than that provided by the linear vector function.

The second type is that of the general vector field in which the characteristic vector is to be regarded as a function of the vector of position. This form can probably be most satisfactorily treated by means of quaternionic functions, as we hope to show at a later point.

In geometry the general vector function represents, as already stated, a very general transformation of space. In consequence of the property it may be employed to present many geometric relations. The transformation is a collineation only in the case of the linear vector function. In the general case only lines through the origin, which remains fixed, are transformed into straight lines which likewise pass through the origin. It will furthermore appear that in the case of vector functions of odd degree the surface defined is closed, while in the case of even degree the surface is entirely confined to the positive semispace.

In purely analytic applications this function can be caused to play an auxiliary rôle, in virtue of the relations between quantics which are uncovered by an investigation of its properties. In the second part this aspect will be given some consideration.

3. *Notation and Definitions*

In the present paper the symbolism plays such an important part that it may be considered as an example of the applications of the method, capable, we hope, of much wider extension. We shall accordingly proceed with a precision of the terms employed.

Let there be given a system of n^{m+1} distinct terms, and let each be represented by a symbol of the form

$$a_{p_1 p_2 \ldots p_{m+1}}$$

in which each of the $m + 1$ suffixes may assume all values from 0 to n, there being obviously the requisite number of such distinct symbols (n^{m+1}). The aggregate of these symbols will be denoted:

$$(a_{\bar{n} \ldots (m+1) \ldots \bar{n}}) = (a_{\bar{n}}{}^{(m+1)}), \tag{1}$$

and will be termed a matrix of the nth order and class $m + 1$.

Then the product of a matrix of class $m + 1$ and one of class 1, will be defined thus:

$$(a_{\bar{n} \ldots (m+1) \ldots \bar{n}})(b_{1\bar{n}}) = (\Sigma a_{\bar{n} \ldots (m) . \bar{n} \bar{p}_1} b_{1 p_1}),$$

being evidently a matrix of class m and order n. We may accordingly again multiply this matrix by a matrix of class 1, obtaining a matrix of class $m - 1$:

$$(a_{\bar{n} \ldots (m+1) \ldots \bar{n}})(b_{1\bar{n}})(b_{2\bar{n}}) = \left(\sum_{p_1=1}^{w} a_{\bar{n} \ldots (m) \ldots \bar{n} p_1} b_{1 p_1} \right)(b_{2\bar{n}}) = \left(\sum_{p_1 p_2=1}^{w} a_{\bar{n} \ldots \bar{n} p_1 p_2} b_{1 p_1} b_{2 p_2} \right) \tag{2}$$

By proceeding in this manner we may successively reduce the class of the matrix

at each multiplication by unity, obtaining after m such multiplications a matrix of class 1:

$$(a_{\bar{n}\ldots(m+1)\ldots\bar{n}})(b_{1\bar{n}})(b_{2\bar{n}})\cdots(b_{m\bar{n}}) = \left(\sum_{p..=1}^{w} a_{\bar{n}p_1\ldots pm}b_{1p_1}b_{2p_2}\cdots b_{m-1pm-1}b_{mpm}\right) \quad (3)$$

where the summation is to be extended as before, each subscript separately being ascribed all the values from 1 to n. We may, and in the sequel frequently shall, denote a vector by a matrix of class 1, in which the constituents are simply the components of the vector. If required, the fundamental vectors of reference may be introduced by means of another matrix of class 1, whose constituents are these unit vectors. Then any vector may be expressed as the product of two matrices of class 1, one of which has numerical constituents, and the other unit vector constituents, thus:

$$\mathfrak{B} = (\boldsymbol{\alpha}_{\bar{n}})(a_{\bar{n}}).$$

In general we shall consider the vector matrix as understood, and denote a vector simply by the corresponding numerical matrix.

Now it is evident that by the process above indicated, we are able to define a matrix of class $m - n$ as a function of n given vectors merely by multiplying a matrix of class m by these n vectors, which may or may not be distinct. In general of course the order in which the multiplication is considered as executed is essential. In the special case when $n = m - 1$, we have thus defined a vector as a function of n given vectors. This is the case specifically treated in this paper, but it may be pointed out, at this stage, that under certain circumstances it may be advisable to consider the more general standpoint. Thus for example we may consider n points in space as defining at each point a set of n vectors, which in turn may by multiplication with a matrix of class m, in the manner indicated, be considered as defining a matrix or vector operator of class $m - n$. It is not difficult to see that in some problems such a device may be of real service.

4. *Homogeneous Vector Function of the* m*th Degree*

We are now in a position, by means of the symbolic resources developed in last section, to set up the expression for the homogeneous vector function of the mth degree, as this term was defined in the first section, as follows:

$$\begin{aligned} \varphi_m(\mathfrak{B}) &= \varphi_m\cdot\{(b_{\bar{n}})(\boldsymbol{\alpha}_{\bar{n}})\} \equiv \varphi_m(b_{\bar{n}}), \\ &= (a_{\bar{n}}{}^{(m+1)})(b_{\bar{n}}\overset{m}{)(}\boldsymbol{\alpha}_{\bar{n}})^m \equiv (a_{\bar{n}}{}^{(m+1)})(b_{\bar{n}})^m. \end{aligned} \quad (4)$$

This expression is evidently a matrix of class 1, and accordingly, as already pointed out, may be regarded as a vector, and functionally related to the given vector as follows:

$$\varphi_m(kb_{\bar{n}}) = k^m\cdot\varphi_m(b_{\bar{n}}),$$

which, by definition, is characteristic of the homogeneous vector function of the mth degree. We shall accordingly employ this notation to develop its properties.

We shall in the first place develop the addition theorems, that is to say, set up the expressions for the sum of the vector functions of a set of vectors, and also the expression for the vector function of a sum of vectors. Each of these relations will be developed first for the special case of $m = 2$, not because the general case presents any additional difficulty, but because in the latter certain features, which are prominent in the former, become more subsidiary, and less interesting.

We shall also determine an alternative form for this function, analogous to, and to be regarded as a generalization of the m-adic of J. W. Gibbs.

5. *First Addition Theorem*

We shall give the proof of this theorem in its most general form, since from this standpoint its essential properties are more sharply defined than would be the case if we made a more special assumption regarding m.

We have then:

$$\varphi_m(b_{1\bar{n}} + b_{2\bar{n}}) = (a_{\bar{n}}^{(m+1)})(b_{1\bar{n}} + b_{2\bar{n}})^m,$$

$$= \sum_{r_1 \ldots r_m = 1}^{n} a_{\bar{n} r_1 \ldots r_m}(b_1 + b_2)_{r_1}(b_1 + b_2)_{r_2} \cdots (b_1 + b_2)_{r_m} \tag{5}$$

where the subscripts outside the brackets are the second, those inside the first subscripts to be ascribed the b's.

Let $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_\sigma^{\alpha_\sigma}$, where each of the q's is a prime. If, as customary, we define the special mth roots of unity to be those mth roots of unity which are not also nth roots of unity, where $n < m$, then the following propositions are well known, or may be readily deduced:

I. There are $m\left(1 - \dfrac{1}{q_1}\right)\left(1 - \dfrac{1}{q_2}\right) \cdots \left(1 - \dfrac{1}{q_\sigma}\right)$ special mth roots of unity.

II. If ω is any special mth root of unity, then ω^2 when τ is prime to m, is likewise a special mth root and all the special mth roots are of this form.

III. If ω is a special mth root, then all the roots are given by the series:

$$\omega^0 \omega^1 \omega^2 \omega^3 \cdots \omega^{m-1}.$$

IV. The sum of all the mth roots is zero, whence it follows immediately that if ω is a special mth root:

$$\sum_{\mu=0}^{m-1} \omega^\mu = 0,$$

V. More generally, if ϵ is any mth root of unity then:

$$\sum_{\kappa=1}^{m} \epsilon^\kappa = 0.$$

VI. It therefore follows immediately that, if ω is any mth special root, then:

$$\sum_{\kappa=1}^{m} (\omega^q)^\kappa = 0,$$

for all values of $q < m$.

VI. The sum of all the special mth roots is 0.

VII. If ω_μ be any special μth root, then:

$$\sum_{\kappa=1}^{\mu\rho} (\omega_\mu{}^q)^\kappa = 0, \tag{6}$$

for all values of q, prime to μ.

We shall now apply these properties of the special mth roots to the problem in hand. Let ω be such a special mth root of unity. Consider the product:

$$(a_{\bar{n}}{}^{m+1})(b_{1\bar{n}} + b_{2\bar{n}}\omega^\kappa)^{1n} = \sum_{r_1 \dots r_m = 1}^{n} a_{\bar{n} r_1 \dots r_m}(b_1 + \omega^\kappa b_2)_{r_1} \cdots (b_1 + \omega^\kappa b_2)_{r_m}. \tag{7}$$

In this product any term which involves b_2 q times, and consequently b_1 $(m - q)$ times, has the factor $\omega^{\kappa q}$, in addition to a vectorial factor which we may reserve for future consideration. If then we form all the distinct products obtained by ascribing to k all values from 1 to m, and take their sum, then in the latter such a term as above indicated will be multiplied by the factor:

$$\sum_{\kappa=1}^{m} (\omega^{\kappa q})$$

which in virtue of the properties of the special mth roots of unity, as above developed, is zero. Accordingly this sum simply reduces to the sum of the vector functions of $b_{1\bar{n}}$ and $b_{2\bar{n}}$, multiplied by m. That is to say:

$$\begin{aligned}
\sum_{\kappa=1}^{m} (a_n{}^{(m+1)})(b_{1\bar{n}} + \omega^\kappa b_{2n}) &= m\{(a_{\bar{n}}{}^{(m+1)})[(b_{1\bar{n}}) + (b_{2\bar{n}})]^m\}, \\
&= m\{\varphi_m(b_{1\bar{n}}) + \varphi_m(b_{2\bar{n}})\}, \qquad (8)\\
&= \sum_{\kappa=1}^{m} (a_{\bar{n}}{}^{(m+1)})(\omega^\kappa b_{1\bar{n}} + b_{2\bar{n}}),
\end{aligned}$$

where the latter relation follows from considerations of symmetry, and requires evidently:

$$\varphi_m(\omega^\kappa b_{1\bar{n}} + b_{2\bar{n}}) = \varphi_m(b_{1\bar{n}} + \omega^\kappa b_{2\bar{n}}).$$

This latter condition is satisfied, for:

$$\begin{aligned}
\varphi_m\{\omega^{m-q}(b_{1\bar{n}} + \omega^q b_{2\bar{n}})\} &= \varphi_m(\omega_1{}^{m-q} b_{1\bar{n}} + b_{2\bar{n}}) = \omega^{m(m-q)} \cdot \varphi_m(b_{1\bar{n}} + \omega_1{}^q b_{2\bar{n}}) \\
&= \varphi_m(b_{1\bar{n}} + b_{2\bar{n}}\omega^q).
\end{aligned}$$

By a repetition of the procedure we obtain:

$$\begin{aligned}
\sum_{\lambda_2=1}^{m}\sum_{\lambda_1=1}^{m} \varphi_m(b_{1\bar{n}} + \omega^{\lambda_1} b_{2\bar{n}} + \omega^{\lambda_2} b_{3\bar{n}}) &= m \cdot \sum_{\lambda_1=1}^{m} (b_{1\bar{n}} + \omega^{\lambda_1} b_{2\bar{n}}) + m^2 \varphi_m(b_{3\bar{n}}), \qquad (9)\\
&= m^2\{\varphi_m(b_{1\bar{n}}) + \varphi_m(b_{2\bar{n}}) + \varphi_m(b_{3\bar{n}})\}.
\end{aligned}$$

It is evident that the general expression is obtained by successive repetitions

of the above process, in the form

$$\sum_{\lambda_p=1}^{m} \cdots \sum_{\lambda_2=1}^{m} \varphi_m \left(b_{1\bar{n}} + \sum_{s=1}^{p} \omega^{\lambda_2} b_{s\bar{n}} \right) = m^{p-1} \cdot \sum_{s=1}^{p} \varphi_m(b_{s\bar{n}}),$$

representing an identical relation between two sums of vector functions of the same order and degree.

It may be pointed out that these are not the only identities which may be obtained by an analogous process.

6. *First Addition Theorem in Case $m = 2$*

As a special case of the above theorem, when $m = 2$, we have since then

$$\omega = -1, \qquad \omega^2 = 1,$$

the following:

$$\varphi_2(b_{1\bar{n}} + b_{2\bar{n}}) + \varphi_2(b_{1\bar{n}} - b_{2\bar{n}}) = 2 \cdot \{\varphi_2(b_{1\bar{n}}) + \varphi_2(b_{2\bar{n}})\}.$$

Now on each side of this equation occurs a sum of two vectors. Consequently if this is regarded as giving an identical expression for the sum of the vector functions of the vectors $(b_{1\bar{n}})$, $(b_{2\bar{n}})$, on reading from right to left, then the application of this same theorem to the sum of the vector functions of $(b_{1\bar{n}} + b_{2\bar{n}})$, $(b_{1\bar{n}} - b_{2\bar{n}})$, on the left must give an expression reducible ultimately to that occurring on the right of the above equation. That in this simple instance this is as a matter of fact the case, is obvious. For we have:

$$(b_{1\bar{n}} + b_{2\bar{n}}) + (b_{1\bar{n}} - b_{2\bar{n}}) = 2b_{1\bar{n}},$$

$$(b_{1\bar{n}} + b_{2\bar{n}}) - (b_{1\bar{n}} - b_{2\bar{n}}) = 2b_{2\bar{n}},$$

and therefore:

$$\begin{aligned}\varphi_2(b_{1\bar{n}} + b_{2\bar{n}}) + \varphi_2(b_{1\bar{n}} - b_{2\bar{n}}) &= \tfrac{1}{2}\{\varphi_2(2b_{1\bar{n}}) + \varphi_2(2b_{2\bar{n}})\}, \\ &= 2\{\varphi_2(b_{1\bar{n}}) + \varphi_2(b_{2\bar{n}})\},\end{aligned}$$

which is the same result as above obtained.

Consider similarly the more general case of this theorem when $m = 2$:

$$\begin{aligned}\varphi_2(b_{1\bar{n}} + b_{2\bar{n}} + b_{3\bar{n}}) + \varphi_2(b_{1\bar{n}} + b_{2\bar{n}} - b_{1\bar{n}}) + \varphi_2(b_{1\bar{n}} - b_{2\bar{n}} + b_{3\bar{n}}) \\ + \varphi_2(b_{1\bar{n}} - b_{2\bar{n}} - b_{3\bar{n}}) = 2^2 \cdot \{\varphi_2(b_{1\bar{n}}) + \varphi_2(b_{2\bar{n}}) + \varphi_2(b_{3\bar{n}})\}.\end{aligned} \tag{10}$$

If we consider this as giving an identity for the vector function sum on the right, then conversely the application of this addition theorem to the sum on the left into which enter the vectors:

$$(b_{1\bar{n}} + b_{2\bar{n}} + b_{3\bar{n}}), \qquad (b_{1\bar{n}} + b_{2\bar{n}} - b_{3\bar{n}}), \qquad (b_{1\bar{n}} - b_{2\bar{n}} + b_{3\bar{n}}), \qquad (b_{1\bar{n}} - b_{2\bar{n}} - b_{3\bar{n}})$$

an expression must be obtained which ultimately reduces to the expression on the right. The reader may readily convince himself that such is the case, though the formal proof is too tedious to be given here.

Finally consider the most general case of the theorem for $m = 2$:

$$\Sigma_2 \varphi_2(\Sigma_1 b_{s\bar{n}}) = 2^{p-1} \cdot \sum_{s=1}^{p} \varphi_2(b_{s\bar{n}}), \tag{11}$$

where the first summation Σ_1, is to be extended over the b's, ascribing b_1 the positive sign and to the remaining b's either the $+$ or $-$ sign, and then extending the summation Σ_2 over all such expressions without repetition, obtained by taking all possible combinations of sign. There are evidently 2^{p-1} such terms.

Now those relations which have been indicated in the preceding simple cases must also hold here. That is to say, the application of the general law obtained by reading the equation from left to right or right to left must ultimately yield the same expression. That such is the case is not self-evident, in fact it gives rise to a theorem, which however does not appear to be of sufficient interest to state explicitly.

7. *Transformation of Matrix Expression*

In the preceding we have based the definition of the vector function operator φ_m on a matrix expression $(a_{\bar{n}}^{(m+1)})$. We shall now transform latter into an expression which is the generalization of the dyadic expression of Gibbs.

Evidently

$$\varphi_m(b_{\bar{n}}) = (a_{\bar{n}}^{(m+1)})(b_{1\bar{n}})^m(\boldsymbol{\varepsilon}_{\bar{n}}) = \sum_{p_1=1}^{\omega} \boldsymbol{\varepsilon}_{p_1}\boldsymbol{\alpha}_{p_1} \cdot \boldsymbol{\beta},$$

where we assume that the matrix $(a_n^{(m+1)})$ represents the vector function φ_m with respect to the set of independent vectors $(\epsilon_{\bar{n}})$, and the vector $(\epsilon_{\bar{n}})(b_{\bar{n}})$ is written $\boldsymbol{\beta}$. Further in this equation we have written:

$$\boldsymbol{\alpha}_{p_1} = (a_{p_1\bar{n}\ldots(m)\ldots\bar{n}})\overset{m-1}{(b_{\bar{n}})}(\boldsymbol{\varepsilon}_{\bar{n}}) = \left(\sum_{p_2=1}^{\omega} \boldsymbol{\varepsilon}_{p_2} a_{p_1p_2\bar{n}(m-1)\cdots\bar{n}}\right)(b_{\bar{n}})^{m-1},$$

$$= \left(\sum_{p_2=1}^{\omega} \boldsymbol{\varepsilon}\boldsymbol{\alpha}_{p_1p_2}\right) \cdot \boldsymbol{\beta},$$

and

$$\begin{aligned} \boldsymbol{\alpha}_{p_1p_2} &= (a_{p_1p_2\bar{n}(m-1)\ldots\bar{n}})\overset{m-2}{(b_{\bar{n}})}(\boldsymbol{\varepsilon}_{\bar{n}}), \\ &= \left(\sum_{p_3=1}^{\omega} \boldsymbol{\varepsilon}\boldsymbol{\alpha}_{p_1p_2p_3}\right) \cdot \boldsymbol{\beta}, \end{aligned} \tag{12}$$

and

$$\boldsymbol{\alpha}_{p_1p_2p_3} = (a_{p_1p_2p_3\bar{n}(m-2)\ldots\bar{n}})\overset{m-3}{(b_{\bar{n}})}(\boldsymbol{\varepsilon}_{\bar{n}}),$$

$$= \left(\sum_{p_4=1}^{\omega} \boldsymbol{\varepsilon}_{p_4}\boldsymbol{\alpha}_{p_1\ldots p_4}\right) \cdot \boldsymbol{\beta},$$

etc., etc.

Finally:

$$\begin{aligned} \boldsymbol{\alpha}_{v_1\cdots p_{m-1}} &= (a_{p_1\ldots p_{m-1}\bar{n}\bar{n}})(b_{\bar{n}})(\epsilon_{\bar{n}}), \\ &= \left(\sum_{p_m=1}^{\omega} \epsilon_{p_m}\boldsymbol{\alpha}_{p_1\ldots p_m}\right) \cdot \boldsymbol{\beta}, \end{aligned} \tag{13}$$

and

$$\mathbf{a}_{p_1\ldots p_m} = \left(\sum_{p_{m+1}=1}^{\omega} \boldsymbol{\varepsilon} a_{p_1\ldots p_{m+1}} \right).$$

Consequently we obtain:

$$(a_{\bar{n}}{}^{(m+1)})(b_{\bar{n}})^{m}(\boldsymbol{\varepsilon}_{\bar{n}}) = \sum_{p_1=1}^{\omega} \boldsymbol{\varepsilon}_{p_1} \left\{ \sum_{p_2=1}^{\omega} \boldsymbol{\varepsilon}_{p_2} \left\{ \sum_{p_3=1}^{\omega} \boldsymbol{\varepsilon}_{p_3} \right.\right.$$

$$\left.\left. \times \left\{ \cdots \left\{ \boldsymbol{\varepsilon}_{p_{m+1}} a_{p_1\ldots p_{m+1}} \right\}_m \cdot \boldsymbol{\beta} \right\}_{m-1} \cdot \boldsymbol{\beta} \cdots \right\} \cdot \boldsymbol{\beta} \right\} \tag{14}$$

Consider now the special case that each of the forms

$$(a_{r_1\bar{n}\ldots(m)\ldots\bar{n}})(b_{\bar{n}})^m, \qquad p_1 = 1 \cdots n,$$

is resolvable into linear factors, so that we have:

$$(a_{p_1\bar{n}\ldots(m)\ldots\bar{n}})(b_{\bar{n}})^m = (c_{p_1 1\bar{n}})(b_{\bar{n}})(c_{p_1 2\bar{n}})(b_{\bar{n}}) \cdots (c_{p_1 m\bar{n}})(b_{\bar{n}}),$$

$$= (\boldsymbol{\gamma}_{p_1 1} \cdot \boldsymbol{\beta})(\boldsymbol{\gamma}_{p_1 2} \cdot \boldsymbol{\beta}) \cdots (\boldsymbol{\gamma}_{p_1 m} \cdot \boldsymbol{\beta}), \tag{15}$$

where

$$\boldsymbol{\gamma}_{p_1 r} = (\mathbf{c}_{p_1 r\bar{n}})(\boldsymbol{\varepsilon}_{\bar{n}}).$$

Then the above expression reduces to the simpler form:

$$\varphi_m(\boldsymbol{\beta}) = (\boldsymbol{\varepsilon}_1 \boldsymbol{\gamma}_{11}\boldsymbol{\gamma}_{12}\boldsymbol{\gamma}_{13} \cdots \boldsymbol{\gamma}_{1m} + \cdots + \boldsymbol{\varepsilon}_n \boldsymbol{\gamma}_{n_1}\boldsymbol{\gamma}_{n_2}\boldsymbol{\gamma}_{n_3} \cdots \boldsymbol{\gamma}_{mn}) \cdot (\boldsymbol{\beta}). \tag{16}$$

This is the direct generalization of Gibbs' Dyadic, and will be termed the m-adic.

8. *Derivation of Alternative Expressions for Vector Function Operator*

In the above we showed that a vector function which was represented by a matrix of the $(m + 1)$th class was as a matter of fact a homogeneous vector function of the mth degree. In this section we shall show conversely that every homogeneous vector function may be expressed as such a matrix.

Let Φ be a vector function operator operating distributively, but not necessarily commutatively, so that we have:

$$\Phi \cdot (\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n) = \Phi(\mathbf{a}_1) + \Phi(\mathbf{a}_2) + \cdots + \Phi(\mathbf{a}_n),$$

each summand being again a vector operator. Whence:

$$\Phi \cdot (\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \cdots + \mathbf{a}_n)^m = \Phi \cdot [\mathbf{a}_1 + \mathbf{a}_2 + \cdots + \mathbf{a}_n]^m,$$

symbolically, where we assume the multinomial on the right to be expanded, the order of the $\boldsymbol{a}$'s being assumed as retained in every case, and then each term in that form operated on by Φ. By making use of the notion of a gap expression, due to Grassmann, we may without loss of generality, replace the operator Φ by an operator Ψ, which is to be assumed as operating commutatively.

Consider an expression into which enter p symbols

$$l_1 l_2 l_3 \cdots l_p,$$

each in a perfectly distinct manner, and which we shall denote $L \cdot (l_1 l_2 l_3 \cdots l_p)$.

Thus of course any algebraic expression into which enter both stated quantities or numerical coefficients and general symbols, is such a gap expression. The number of gaps in such an expression is that number of general symbols which it is necessary to define before the expression for the purpose in hand, is determined. Thus the number may vary, as in the following case of a polynomial in a set of variables. If the form of the polynomial alone be in question, the gaps are the constant coefficients, the variables being left undefined. If however the value of the polynomial be considered, then in addition it is necessary to define the values of the variables.

Then we define as the product of L. and a set of p values (or at least domains of values), that value or expression which is obtained by replcaing all the gaps $l_1 l_2 l_3 \cdots l_p$, by the given values in all possible distinct combinations without repetition, and dividing the sum of the results by their number. It is then evident that the order of the operands is immaterial, that is to say, the operation is commutative.

Replace now $\Sigma\Phi\cdot(\beta_{r_1}{}^{a_1'}\beta_{r_2}{}^{a_2'}\cdots\ \beta_{r_s}{}^{a_s'})$ (17), where the indices $a_1' a_2' a_3' \cdots a_s'$, are to be considered as constant for each sum, and the summation is to be extended over all possible sequences of the β's, by the gap product operator Ψ. Operating on the symbolic product $(\beta^{a_1}\beta^{a_2}\cdots\ \beta_n{}^{a_n})$ in which now the factors may be considered as commutative, multiplied of course by the multinomial coefficient

$$\frac{m!}{a!\,a!\cdots a!}$$

Then we may write for above expression:

$$\Psi\cdot(\beta_1+\beta_2+\beta_3\cdots+\beta_n)^m = \Psi\cdot[\beta_1+\beta_2+\beta_3\cdots+\beta_n]^m,$$

where the expression on the right is to be assumed as expanded by the multinomial theorem, and then each term operated on by Ψ. It then follows from the property characteristic of the homogeneous vector operator that

$$\Psi\cdot(c_1\boldsymbol{a}_1+c_2\boldsymbol{a}_2+c_3\boldsymbol{a}_3+\cdots+c_n\boldsymbol{a}_n)^m = \Psi\cdot[c_1\boldsymbol{a}_1+c_2\boldsymbol{a}_2+\cdots+c_n\boldsymbol{a}_n]^m, \quad (18)$$
$$= (\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\cdots\ \boldsymbol{A}_\omega)(c_{\bar{n}})^m,$$

in the ordinary notation for an n-ary m-ic. Here the coefficients $\boldsymbol{A}_{\omega_1\omega_2\cdots\omega_n}$, (18$a$) etc., are the products $\Psi\cdot(\beta_1{}^{\omega_1}\beta_2{}^{\omega_2}\cdots\ \beta_n{}^{\omega_n})$ (19) as above defined. Now when we write for the above form the matrix representation,

$$(\boldsymbol{A}_{\bar{n}}{}^{(m)})(c_{\bar{n}})^m,$$

in which the coefficients of the first matrix are vectors (or, as we also consider them, simply extensive magnitudes), and which is to be considered as symmetric, we have obtained an expression precisely equivalent to that which we have hitherto made the basis of our presentation. Thus if we assume, as may be done without loss of generality, that the matrix expression referred to the set of vectors

$(\mathfrak{a}_n)$ be $(a_{\bar{n}}{}^{(m+1)})$, and symmetric with respect to all the subscripts except the first, then:

$$(a_{\bar{n}(\omega_1)\dots(\omega_n)})(\mathfrak{a}_{\bar{n}}) = \boldsymbol{A}_{\omega_1\dots\omega_n}, \text{ etc.,} \tag{20}$$

where $a_{\bar{n}(\omega_1)\dots(\omega_n)}$ signifies a term in which ω_1 of the subscripts are equal to 1, ω_2 equal to 2, etc., all terms of this common form being by assumption equal.

9. *Products of Extensive Magnitudes*

We may also obtain the above results by considering products of extensive magnitudes, as follows. Write

$$\begin{aligned} \boldsymbol{A} &= (a_{\bar{n}}{}^{(m+1)}) = \sum_{p=1}^{\omega} a_{p_1 p_2 \dots p_{m+1}} \boldsymbol{\varepsilon}_{1p_1} \boldsymbol{\varepsilon}_{2p_2} \cdots \boldsymbol{\varepsilon}_{m+1 p_{m+1}} \\ \boldsymbol{B} &= (b_{\bar{n}}) = \sum_{p_1=1}^{\omega} b_{p_1} \Big| \boldsymbol{\varepsilon}_{1p_1} = \sum_{p_2=1}^{\omega} b_{p_2} \Big| \boldsymbol{\varepsilon}_{2p_2} = \cdots \end{aligned} \tag{21}$$

where $|\boldsymbol{\varepsilon}_p$ denotes as usual the complement of $\boldsymbol{\varepsilon}_p$ and accordingly $\boldsymbol{\varepsilon}_r | \boldsymbol{\varepsilon}_s = 1$, or $= 0$, according as $r \neq s$, or $r = s$. Then evidently the product is of the same form as the products of the corresponding matrices, as above defined.

Suppose that we multiply $\boldsymbol{A}$ by $\boldsymbol{B}$ μ times, where the class of $\boldsymbol{A}$ (which is equal, by definition, to the class of the corresponding matrix) is $\geq \mu$. Then we have for

$$\boldsymbol{A} \cdot \boldsymbol{B}^{\mu}$$

an algebraic expression, or homogeneous form, of the μth degree in n variables, whose coefficients are extensive magnitudes of class $m - \mu + 1$. We may treat this form as equivalent to a system of forms with extensive coefficients of lower class, in various ways, according to the requirements of the problem. The spirit of the method consists then in this, that a theorem which holds for any form, or system of forms, with coefficients which are extensive magnitudes of any class, may be readily, with certain limitations, which will become obvious in the sequel, generalized for all such forms. By this means the invariant relations are exhibited most naturally. Before investigating this side of the subject it is however necessary to develop to some extent the theory of determinants of the pth class.

10. *Transformation of Matrix Expression*

Suppose that with reference to a set of n independent vectors $(\mathfrak{a}_{\bar{n}})$, the matrix of the vector function operator be $(a_{\bar{n}}{}^{(m+1)})$. It is required to determine the transformation which this matrix undergoes when we pass to a second set of independent vectors $(\boldsymbol{\beta}_{\bar{n}})$, related to the former by the relation:

$$(\mathfrak{a}_{\bar{n}}) = (d_{\bar{n}\bar{n}})^{-1}(\boldsymbol{\beta}_{\bar{n}}). \tag{22}$$

Evidently

$$\begin{aligned} \Phi \cdot (\boldsymbol{K}) &= (\mathfrak{a}_{\bar{n}})(a_{\bar{n}}{}^{(m+1)})(k_{\bar{n}})^m, \\ &= (\boldsymbol{\beta}_{\bar{n}})(d_{\bar{n}\bar{n}})^{-1} : (a_{\bar{n}}{}^{(m+1)}) : (d_{\bar{n}\bar{n}})(k_{\bar{n}}')^m, \\ &= (\boldsymbol{\beta}_{\bar{n}})(b_{\bar{n}}{}^{(m+1)})(k_{\bar{n}}')^m. \end{aligned} \tag{23}$$

Consequently

$$(b_{\bar{n}}{}^{(m+1)}) = (d_{\bar{n}\bar{n}}{}')^{-1}(a_{\bar{n}}{}^{(m+1)})(d_{\bar{n}\bar{n}})^m,$$

(24)
$$= (D_{\bar{n}\bar{n}}{}')(a_{\bar{n}}{}^{(m+1)})(d_{\bar{n}\bar{n}})^m,$$

$$= \left(\sum_{p=1}^{\omega} a_{p_{m+1}\ldots p_1} d_{p_1\bar{n}} d_{p_2\bar{n}} \cdots d_{p_m\bar{n}} D_{p_{m+1}\bar{n}}\right)$$

The result is evidently again a matrix of class $(m+1)$, in which the second suffixes of the d's and of D are the determining suffixes.

II. DETERMINANTS OF THE P–TH CLASS

1. *Introductory*

Let $\boldsymbol{e}_{1\bar{n}}\boldsymbol{e}_{2\bar{n}}\boldsymbol{e}_{3\bar{n}} \cdots \boldsymbol{e}_{p-1\bar{n}}$, be $p-1$ unrelated sets of independent extensive magnitudes, or alternate numbers, which accordingly, for all values of the first subscripts satisfy the relations:

$$[\boldsymbol{e}_{r\lambda_1}\boldsymbol{e}_{r\lambda_2}\boldsymbol{e}_{r\lambda_3} \cdots \boldsymbol{e}_{r\lambda_4}] = 0, \text{ when an element is repeated,}$$

$$= \pm 1, \text{ when no element is repeated,}$$

the $+$ or $-$ sign being ascribed according as the number of inversions of the second subscripts is even or odd.

Then an extensive magnitude of the $(p-1)$th class and nth order is defined as the product:

$$\boldsymbol{A} = (a_{\bar{n}}{}^{(m+1)})(\boldsymbol{e}_{1\bar{n}})(\boldsymbol{e}_{2\bar{n}}) \cdots (\boldsymbol{e}_{p-1\bar{n}}).$$

The product of n such magnitudes is evidently a scalar, and will be termed the determinant of the matrix $(\boldsymbol{A}_{\bar{n}}) = (a_{\bar{n}\bar{n}\ldots(p-1)\ldots\bar{n}})$, determined by these magnitudes, with respect to the first suffix. Thus

(25)
$$|a_{\bar{n}\ldots(p)\ldots\bar{n}}| = [\boldsymbol{A}_1\boldsymbol{A}_2 \cdots \boldsymbol{A}_n] = \prod_{q=1}^{\omega}\left(\sum_{\kappa=1}^{n} a_{q\kappa_1\ldots\kappa_{p-1}}\boldsymbol{e}_{1\kappa_1}\boldsymbol{e}_{2\kappa_2} \cdots \boldsymbol{e}_{p-1\kappa_{p-1}}\right).$$

In a similar manner may be defined the determinants of the matrix with respect to the remaining suffixes. Each of these determinants will consist of the algebraic sum of products of the form:

$$\pm\, a_{\alpha_1\ldots\alpha_p} a_{\beta_1\ldots\beta_p} \cdots a_{\nu_1\ldots\nu_p},$$

there being in each product n factors, supposed ordered according to the suffix of reference, in this case the first, and every value from 1 to n being represented in case of each suffix, once and once only. The $+$ or $-$ sign is to be taken in the case of any given product, according as the number of inversions of the suffixes when, as stated, the suffixes of reference are ordered naturally, is even or odd. The summation is then to be extended over all possible products of this type, without repetition.

It is then evident that the products which enter into the expressions of any two distinct determinants are, except possibly for sign, the same in each case. We now propose to investigate the change of sign which any particular term undergoes when we pass from a determinant with respect to any given suffix to that with respect to any other.

Let the first determinant be that with respect to the σth suffix, the second that with respect to the τth. The sign of any particular term in the former case will be determined by the number of inversions by which each of the remaining suffixes is affected, when the σth suffixes are ordered naturally; while in the latter case it will be determined by the corresponding sum of inversions when the τth suffixes are ordered naturally.

Denote former sum thus:

$$I_\sigma = q_1 + q_2 + q_3 + \cdots + q_{\sigma-1} + q_{\sigma+1} + \cdots + q_\tau + \cdots + q_p.$$

Now if $q_1'q_2'$, etc., be the corresponding number of inversions (increased or decreased by an even number, which does not affect the present question, which is simply to determine whether I is even or odd) in for the same term, in the case of the second determinant, i.e. when the suffixes are ordered naturally, then:

$$q_r' = q_r + q_\tau, \text{ etc.},$$

$$q_\sigma' = q_\tau.$$

Consequently $I_\tau = I_\sigma + (p - 2)q_\tau$, and therefore the sign of this term in the two suffixes differs by

$$(-1)^{pq\tau} = (-1)^{pq\sigma}.$$

Whence:

Theorem.—Determinants of even class are independent of the suffix of reference. In the case of determinants of odd class, the determinants are all distinct, the only term which has the same sign for all such determinants being the principal term $a_{1\ldots1}a_{2\ldots2}a_{3\ldots3}\cdots a_{n\ldots n}$.

From these facts alone, it will be inferred that the determinants of the matrices of forms of even degree, which of course are determinants of even class, will play a much more important rôle than the determinants of forms of odd degree. We shall see indeed that they are invariants of these forms. On the other hand, for systems of forms, whose matrix will have one of the suffixes distinguished from the remainder, determinants of odd class will be significant, and will indeed also be found to have the invariant property. The theory of forms of even degree will, from this standpoint, be seen to differ essentially from that of forms of odd degree. This is in harmony with the results of other methods of investigation. However it will be seen later that the theory of forms of odd degree can also be treated by this method. It will however first be necessary to develop the theory of determinants of even class, which we now proceed to do.

We shall first determine certain algebraic properties of these determinants, following immediately from their definition in terms of alternate numbers, as

already given. Then we shall proceed to the consideration of the multiplication of determinants, and the relation which exists between the determinant of the product of matrices and the product of the determinants of the separate factors. From this will flow readily the invariant property of determinants of even forms.

2. *Fundamental Properties of Determinants of Even Class*

We shall set up the following:

Definition.—The matrix of class $(p - 1)$ obtained by ascribing to the rth suffixa certain value, π_1 say, and ascribing to the remaining suffixes, as before the values 1 to n, will be termed a first face of the matrix, and denoted $(r_1 |^{\pi_1})$.

Similarly a first face of this matrix will be termed a second face of the original matrix, and so forth.

Theorem.—If in the case of matrices of even class, the significance of two faces with respect to the same suffix be interchanged, then the absolute value of the determinant is unchanged, but it changes sign.

By this is simply meant, that if we assume the determinant as expressed in terms of the general symbols $a_{\lambda_1 \ldots \lambda_p}$, then we shall obtain two quantities when we replace these symbols by definite values, first so that the symbols of a pair of faces with respect to the same suffix assume a given set of values, and secondly when these faces assume these same values, but in the reverse order. Then the theorem states that the absolute values of these determinants will be the same in each case, but that they will have opposite signs. The theorem is now evident from this explicit statement. For such an interchange is equivalent to an interchange of two alternate numbers in each product, which effects a change of sign. Thus interchange of the values to be ascribed to the corresponding letters of the faces $(r |^{\pi_1})$ $(r |^{\pi_2})$, is equivalent to the interchange of the alternate numbers $\boldsymbol{e}_{r\pi_1}$ and $\boldsymbol{e}^{r\pi_2}$ throughout, which evidently changes the sign of every term, provided the quantities thus attributed to the letters are extensive magnitudes of even class (under this term of course being included those of zero class, that is, ordinary numbers.) The permutation of such numbers does not affect the sign of the product.

If the face be one with respect to the suffix of reference, then the interchange of the values of two such faces, in the case of determinants of even class evidently also effects a change of sign, as may be inferred either from the equivalence of the different determinants, or from the fact that interchange of two such faces is equivalent to the interchange of $p - 1$ sets of alternate numbers, which effects a change in sign.

When p is odd, however, the latter argument shows that in this case no change in sign is effected. Consequently we have:

Theorem.—In the case of determinants of odd class, interchange of two faces which are not faces of reference, effects a change in sign, interchange of two faces of reference leaves sign unchanged.

3. *Note on Determinants of Odd Class*

It has already been pointed out that the determinants of matrices of odd class are in general all distinct. We propose to investigate the nature of the sum of all these determinants, confining ourselves in the first place to cubic determinants.

Adopt the following symbolism: For the sum of the terms, which are such that when the first suffix is taken as suffix of reference, have an even number of inversions of the second suffix, and likewise an even number of inversions of the third suffix, write $T(ree)$, and similarly for the other terms $T(reo)$, $T(roe)$, $T(roo)$. Then we have:

$$D = T(ree) - T(reo) - T(roe) + T(roo),$$

$$= T(eee) - T(eeo) - T(eoe) + T(eoo),$$

as it may also be written more symmetrically. Further,

$$D_1 + D_2 = 2T(eee) - 2T(eeo),$$

$$D_1 + D_3 = 2T(eee) - 2T(eoe),$$

and

$$D_1 + D_2 + D_3 = 3T(eee) - T(eeo) - T(eoe) - T(oee).$$

This result may be readily generalized. These sums are then completely determined by the matrix in question, without ambiguity, but it may be easily seen that they are not invariants of the corresponding forms.

4. *Transformation of Forms*

Consider the n-ary p-ic $(a_{\bar{n}}{}^{(p)})(x_{\bar{n}})$ whose matrix is $(a_{\bar{n}\ldots(p)\ldots\bar{n}})$. If for the variables $x_{\bar{n}}$ be substituted the variables $y_{\bar{n}}$, related to the former by the equation $x_{\bar{n}} = y_{\bar{n}}\cdot\tau_{\bar{n}\bar{n}}'$, then the quantic is transformed into the n-ary p-ic in these y's, given by:

$$(a_{\bar{n}\ldots(p)\ldots\bar{n}}) : (\tau_{\bar{n}\bar{n}})(y_{\bar{n}})^p,$$

whose matrix is

$$(a_{\bar{n}\ldots(p)\ldots\bar{n}}) : (\tau_{\bar{n}\bar{n}})^p = \left(\sum_{q=1}^{n} a_{q_1 q_2\ldots q_p}\tau_{q_1\bar{n}}\cdots\tau_{q_p\bar{n}}\right), \tag{27}$$

where the determining suffixes are now the second suffixes of the τ's.

The same relations hold of course in the case of a system of m n-ary forms, whose matrix is $(a_{\bar{m}\bar{n}\ldots(p)\ldots\bar{n}})$ and whose transformed matrix is $(a_{\bar{m}\bar{n}\ldots(p)\ldots\bar{n}}) : (\tau_{\bar{n}\bar{n}})^p$.

More generally, suppose in the above quantic that the variables $x_{\bar{n}}$ be replaced by quantics not necessarily linear, in the variables $y_{\bar{n}}$. That is, we write:

$$(x_{\bar{n}}) = (\tau_{\bar{n}}{}^{(\sigma+1)})(y_{\bar{n}})^{\sigma}.$$

Then the original matrix is transformed into an n-ary $(p\cdot\sigma)$ic. Its matrix is given by

$$(a_{\bar{n}\ldots(p)\ldots\bar{n}}) : (\tau_{\bar{n}\ldots(\sigma)\ldots\bar{n}})^p = \left(\sum_{q_1\ldots q_p=1}^{n} a_{q_1 q_2\ldots q_p}\tau_{q_1\bar{n}\ldots\bar{n}}\cdots\tau_{q_p\bar{n}\ldots\bar{n}}\right), \tag{28}$$

in which the last σ suffixes of the τ's are now the determining suffixes.

As before, the same relations hold if, instead of considering a single quantic, we consider a system of such quantics, the resulting expression being obtained by merely adding another subscript to the a's.

Further it may be convenient to generalize the relation between the original variables, and the substituted variables in a different manner. Thus instead of substituting for the former a single definite system, we may consider them as substituted by a set of such systems, thus:

$$(x_{\bar{n}}) = (\tau_{\bar{n}}{}^{(\sigma+\mu)})(y_{\bar{n}}),$$

which, of course is merely equivalent to considering, and grouping in a single expression, the various equations determined by a set of substitutions. Nevertheless such a treatment is on occasion valuable. The set of forms, thus obtained, is given by the expression:

$$(a_{\bar{n}\ldots\bar{n}}) : (\tau_{\bar{n}\ldots(\sigma)\bar{n}\bar{n}\ldots(\mu)\ldots\bar{n}})^{p}. \tag{30}$$

5. *Products of Determinants of Higher Class*

(1) First expression for product. Let

$$|a_{\bar{n}\ldots\bar{n}}|_1 = \prod_{r=1}^{n} \{(a_{r\bar{n}\ldots\bar{n}})(\mathfrak{n}_{1\bar{n}})(\mathfrak{n}_{2\bar{n}}) \cdots (\boldsymbol{\varepsilon}_{p\bar{n}}),$$

$$|b_{\bar{n}\ldots\bar{n}}|_1 = \prod_{r=1}^{n} \{(b_{r\bar{n}\ldots\bar{n}})(\mathfrak{n}_{1\bar{n}})(\mathfrak{n}_{2\bar{n}}) \cdots (\mathfrak{n}_{\pi\bar{n}}),$$

each being the determinant of the corresponding matrix with respect to the first suffix. Then the expression:

$$|a_{\bar{n}\ldots\bar{n}}|_1 \cdot |b_{\bar{n}\ldots\bar{n}}|_1 = \sum_{r=1}^{n} \{(a_{r\bar{n}\ldots\bar{n}}b_{r\bar{n}\ldots\bar{n}})(\boldsymbol{\varepsilon}_{1\bar{n}}) \cdots (\boldsymbol{\varepsilon}_{p\bar{n}})(\mathfrak{n}_{1\bar{n}}) \cdots (\mathfrak{n}_{\pi\bar{n}})\} \tag{31}$$

for the product is obtained by combining corresponding pairs of factors in the two determinants into a single factor of the resulting determinant; the special selection of pairs in this case being those which are characterized by the same values of the first suffix. Any other combination of factors might also have been adopted. The different expressions, so obtained, have of course all the same value, except possibly for sign. The class of the product, expressed in this manner as a determinant, is $p + \pi + 1$. The process is evidently applicable to determinants of both odd and even class, in which respect it differs from the second expression for the product, next to be considered, which is applicable only in the case of determinants of even class in its full generality.

(2) Second expression for product.

This is the expression which will be of most value in the sequel.

Let $\mathbf{E}_n$ be a system of extensive magnitudes of odd class so that:

$$\mathbf{E}_r \cdot \mathbf{E}_s = - \mathbf{E}_s \cdot \mathbf{E}_r.$$

Further if $(\mathbf{E}_{\bar n}) = (b_{\bar n \ldots \bar n})(\mathfrak{n}_{1\bar n}) \cdots (\eta_{\pi \bar n})$, where the $\mathfrak{n}$'s are assumed to be simple extensive magnitudes, then the product of the $\mathbf{E}$'s given by

$$[\mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_n] = |b_{\bar n \ldots \bar n}|,$$

is a determinant of even class.

More generally, on one hand:

$$\prod_{r=1}^{\omega} \{(a_{\bar n \ldots \bar n})(\mathbf{E}_{1\bar n})(\boldsymbol{\varepsilon}_{2\bar n}) \cdots (\boldsymbol{\varepsilon}_{p\bar n})\} = \prod_{r=1}^{\omega} \{(a_{r\bar n \ldots \bar n}) : (b_{\bar n \ldots \bar n})(\mathfrak{n}_{1n}) \cdots (\boldsymbol{\varepsilon}_{1n}) \cdots\}$$

$$= \prod_{r=1}^{\omega} \{(a_{r\bar n \ldots q} b_{q\bar n \ldots \bar n})(\mathfrak{n}_{1\bar n}) \cdots (\mathfrak{n}_{\pi \bar n})(\boldsymbol{\varepsilon}_{1\bar n}) \cdots (\boldsymbol{\varepsilon}_{p\bar n})\}, \quad (32)$$

and on other hand:

$$= \prod_{r=1}^{\omega} \{(a_{r\bar n \ldots \bar n})(\boldsymbol{\varepsilon}_{1\bar n})(\boldsymbol{\varepsilon}_{2\bar n}) \cdots (\boldsymbol{\varepsilon}_{p\bar n})\} [\mathbf{E}_1 \mathbf{E}_2 \cdots \mathbf{E}_n],$$

$$= |a_{\bar n \ldots \bar n}| \cdot |b_{\bar n \ldots \bar n}|.$$

Consequently we are able to express the product of two determinants of classes $p + 1$ and $\pi + 1$, where one at least of the determinants is of even class, as a determinant of class $p + \pi$.

Consider now the determinant of the matrix of the transformed quantic:

$$(a_{\bar n \bar n \ldots (p) \ldots \bar n}) : (b_{\bar n \ldots (\sigma) \ldots \bar n})^p = \left(\sum_{q=1}^{\omega} a_{q_1 q_2 \ldots q_p} b_{q_1 \bar n \ldots \bar n} b_{q 2\bar n \ldots \bar n} \cdots b_{q p \bar n \ldots \bar n}\right) \quad (34)$$

(where we shall assume the matrix in the b's to be even) taken, for simplicity, with respect to the first suffix. Then by preceding, the determinant

$$\prod_{r=1}^{\omega} \left\{ \left(\sum_{r=1}^{\omega} a_{r\bar n \ldots \bar n} b_{r q \ldots \bar n}\right) : (b_{\bar n \ldots \bar n})^{p-1} \cdot (\boldsymbol{\varepsilon}_{1\bar n}) \cdots (\boldsymbol{\varepsilon}_{p-1 \bar n})(\mathfrak{n}_{1\bar n}) \cdots (\mathfrak{n}_{\sigma-1 \bar n})(\quad) \cdots \right\} \quad (35)$$

(where it may be recalled that the first suffixes of the b's are to be considered as connected with the remaining suffixes of the a's in succession) is equal to:

$$\prod_{q=1}^{\omega} \left\{ \left(\sum_{r=1}^{\omega} a_{r\bar n \ldots \bar n} b_{r q \bar n \ldots \bar n}\right) (\boldsymbol{\varepsilon}_{1\bar n}) \cdots (\boldsymbol{\varepsilon}_{p-1 \bar n})(\mathfrak{n}_{1\bar n}) \cdots (\mathfrak{n}_{\sigma-1 \bar n}) \right\} \cdot |b_{\bar n \ldots \bar n}|^{p-1}. \quad (36)$$

If now the first of the above determinants be even, which implies that the determinant in the a's is even, then it will be equal to the determinant of its matrix with respect to any other suffix. On this supposition above determinant will be equal to:

$$\prod_{r=1}^{\omega} \{(a_{\bar n r \bar n \ldots (p-2) \ldots \bar n}) : (b_{\bar n \ldots \bar n})(\boldsymbol{\varepsilon}_{2\bar n}) \cdots (\boldsymbol{\varepsilon}_{p-1 \bar n})(\mathfrak{n}_{1\bar n}) \cdots (\mathfrak{n}_{\sigma-1 \bar n})\}$$

$$= \prod_{r=1}^{\omega} \{(a_{\bar n r \bar n \ldots \bar n})(\boldsymbol{\varepsilon}_{1\bar n})(\boldsymbol{\varepsilon}_{2\bar n}) \cdots (\boldsymbol{\varepsilon}_{p_1 \bar n})\} \cdot |b_{\bar n \ldots \bar n}|. \quad (37$$

Consequently the determinant of the transformed matrix, as above defined, is, under the assumption that both the original matrix and the matrix of transforma-

tion are of even class:

$$|a_{\bar{n}\ldots\bar{n}}| \cdot |b_{\bar{n}\ldots\bar{n}}|^p.$$

Therefore:

Theorem.—The determinant of any even p-ic is an invariant with respect to any transformation whose matrix is of even class, that is to say, whereby the original variables are substituted by quantics of odd degree.

A particular case is that of linear transformation, which is that generally employed in geometry. The advantage of the present method is that the more general transformation indicated presents substantially no additional difficulties, while the scope of the theory is greatly increased, as will, for instance, become evident when Tschirnhausen's transformation is considered. It is further immediately evident that the weight of these invariant determinants is equal to the degree of the quantic.

6. *Example of Binary Quartic*

Consider the binary quartic

$$(abcde \between x_{\bar{2}})^4 \equiv (a_{\bar{2}\bar{2}\bar{2}\bar{2}})(x_{\bar{2}})^4,$$

the former of these two expressions being the familiar one classical since Cayley, while the latter alternative is in harmony with the symbolism employed in this paper.

The value of its determinant is found to be

$$\begin{aligned} |a_{2222}| &= a_{1111}a_{2222} - a_{1112}a_{2221} + a_{1221}a_{2112} \\ &\qquad - a_{1121}a_{2212} + a_{1122}a_{2211} \\ &\qquad - a_{1211}a_{2122} + a_{1212}a_{2121} \\ &\qquad - a_{1222}a_{2111} \\ &= ae \qquad - 4bd \qquad + 3c^2. \end{aligned}$$

This, by our general theorem, is an invariant of the quartic with respect to any transformation of even class. In the particular case of linear transformation this fact, under somewhat different form, has been known for a long time, this being in fact the invariant generally designated by the symbol I. It furthermore follows from our general considerations that its weight is two.

From the above binary quartic we derive the ternary sextic by multiplying through by ζ^2. In the determinant of this sextic, all the terms which do not involve exactly 2 suffixes of value 3, are zero. Consequently in the expansion of this determinant, any summand in which all three factors are not of this description, will be zero. Now, if we denote the number of suffixes equal respectively to 1 and 2, in the first and second factors of such a summand by n_1', n_2''; n_1', n_2''; n_1', n_2''; then it is evident that the following equations must be satisfied

by positive integral values of the n's:

$$\begin{aligned} n_1' + n_2' &= 4, \\ n_1'' + n_2'' &= 4, \\ n_1''' + n_2''' &= 4; \end{aligned} \qquad \begin{aligned} n_1' + n_1'' + n_1''' &= 6, \\ n_2' + n_2'' + n_2''' &= 6, \end{aligned} \qquad \underbrace{\left.\begin{matrix} n_1'n_2'' \\ n_1'n_2'' \\ n_1'n_2'' \end{matrix}\right\}}_{6''} = 4.$$

These equations are evidently consistent, and if, for example, the first three, and one of the last two are satisfied, the remaining equation is likewise satisfied. The diagram on the right merely summarizes the relations, indicating that the sum of the n's in rows is 4, in columns 6. The following are all the possible solutions:

$$\begin{matrix} 40 & 40 & 31 & 31 & 22 \\ 22 & 13 & 22 & 31 & 22 \\ 04 & 13 & 13 & 04 & 22. \end{matrix}$$

Their interpretation in terms of the coefficients is evidently:

$$\begin{matrix} a & a & b & b & c \\ c & d & c & b & c \\ e & d & d & e & c \end{matrix}$$

each multiplied possibly by a numerical factor, whose value is, as a matter of fact, seen from the following to have in each case the value 1/15:

$$\begin{aligned} \left(\frac{4}{4}\right) a &= \left(\frac{6}{24}\right)(40) \quad \text{or} \quad (40) = 1/15\, a \\ \left(\frac{4}{3}\right) b &= \left(\frac{6}{23}\right)(31) \quad \text{or} \quad (31) = 1/15\, b \\ \left(\frac{4}{2}\right) c &= \left(\frac{6}{22}\right)(22) \quad \text{or} \quad (22) = 1/15\, c \qquad (38) \\ \left(\frac{4}{1}\right) d &= \left(\frac{6}{21}\right)(13) \quad \text{or} \quad (13) = 1/15\, d \\ \left(\frac{4}{0}\right) e &= \left(\frac{6}{20}\right)(04) \quad \text{or} \quad (04) = 1/15\, e \end{aligned}$$

Now it is evident that if this ternary sextic be transformed by any substitution of the variables which leaves the third variable ζ unchanged, then it will pass into a sextic of exactly the same form, bearing the same relation to the corresponding transformed quartic, as the original sextic bore to the original quartic. The modulus of transformation of the sextic is equal to that of the quartic.

Further, the determinant of the sextic being an invariant of weight six this syncopated determinant is an invariant of weight six of the quartic, provided of course that it does not vanish identically. The latter however is readily seen not to be the case, even without determining the actual numerical coefficients of the several terms in this determinant, since it involves terms, of the same form, in odd number. By the preceding it will then consist of a sum of numerical multiples of ace, ad^2, bcd, b^2e, c^3, and the complete solution then requires the determination of these numerical coefficients in the above determinant, which is to be assumed as symmetric. In the next section this problem will be attacked, in a more special case. In the meantime we shall anticipate the result sufficiently to state that the invariant so obtained is

$$ace + 2bcd - ad^2 - b^2e - c^3.$$

It is in fact the invariant which has been designated by the symbol J.

These two invariants, I, J, are the only two independent invariants of the quartic, as is known from other considerations, a fact which forms an illustration of a principle to be developed at a later stage. If we attempt to derive another determinant, say by multiplying the quartic through by $\zeta\eta$, obtaining a quarternary sextic, there results the following. In the first place each summand in this determinant will have four factors, and will be zero unless each factor does not contain one and only one subscript of value 3 and 4. Using same symbolism and diagram as before we shall have accordingly:

$$\underbrace{\left.\begin{array}{l} n_1' + n_2' \\ n_1'' + n_2'' \\ n_1''' + n_2''' \\ n_1^{iv} + n_2^{iv} \end{array}\right\}}_{6} = 4$$

which would require $4\cdot 4 = 2\cdot 6$, and consequently the equations are inconsistent, and there is no summand in the determinant of this form. This is a special case of the following:

Theorem.—In order that the κ-ary π-ic, derived from the q-ary p-ic by the process above indicated of addition of auxiliary variables by multiplication, have a determinant in which all the summands are not zero, it is necessary that the relation

$$\kappa\cdot p = \pi\cdot q$$

be satisfied.

The truth of this theorem is evident from the above special cases, since their generalization involves no new principles. In the case under discussion, we have, since $p = 4$, $q = 2$, for any such derived κ-ary π-ic:

$$2\cdot\kappa = \pi.$$

This condition is satisfied by the ternary sextic already investigated in detail,

and furthermore this sextic or ternary form is the only one that satisfies it. The next quantic which also satisfies this condition is a quaternary octavic, which may be derived either by multiplication by $\zeta^3\eta$ or $\zeta^2\eta^2$.

7. *Coefficient of a Given Term in a Symmetric Determinant*

Consider any term in the expansion of a symmetric determinant of the second class and order n. It will consist of the product of n letters each with 2 subscripts. It is required to find the coefficient of such a term in the expression for the determinant.

Let 1 occur in such a factor as first suffix in combination with p_1 as second suffix. Then p_1 as first suffix in combination with p_2 as second suffix, and so on. Write down the resulting sequence as a cycle of a substitution:

$$1p_1p_2p_3 \cdots p_\sigma.$$

If this cycle does not exhaust all the numbers from 1 to n, repeat the operation as often as required to do so.

Then we say:

Theorem.—The coefficient of this term is 2^κ, where κ is the number of cycles of order ≥ 2, with either $-$ or $+$ sign, according as the number of inversions is odd or even.

(The cycle qpq is termed of order 2.)

If two terms are equal, without being identical, they can differ only in the order in which one or more of their cycles is to be taken; that is to say according as we consider the operation of substitution to be from first to second suffix or inversely. Since each cycle may be reversed independently of the remainder, the number of such equal terms is obtained by considering the number of all possible such pairs of cycles. Furthermore each of these terms has the same sign in the expansion of the determinant. The theorem consequently follows.

In the case of determinants of higher class this method is evidently inapplicable.

8. *Differential Invariants and Covariants*

(1) Hessian.—Let

$$d^2U = \left(\frac{\partial^2 U}{\partial x_{\bar{n}} \partial x_{\bar{n}}}\right)(dx_{\bar{n}})^2$$

and let the variables be transformed as follows:

$$dx_{\bar{n}} = (\tau_{\bar{n}\bar{n}})(dy_{\bar{n}}),$$

so that

$$d^2U = \left(\frac{\partial^2 U}{\partial x_{\bar{n}} \partial x_{\bar{n}}}\right) : (\tau_{\bar{n}\bar{n}})^2 (dy_{\bar{n}})^2$$

$$= \left(\frac{\partial^2 U}{\partial y_{\bar{n}} \partial y_{\bar{n}}}\right)(dy_{\bar{n}})^2,$$

where

$$\left(\frac{\partial^2 U}{\partial y_{\bar n}\partial y_{\bar n}}\right) = \left(\frac{\partial^2 U}{\partial x_{\bar n}\partial x_{\bar n}}\right)(\tau_{\bar n\bar n})^2,$$

and consequently

$$\left|\frac{\partial^2 U}{\partial y_{\bar n}\partial y_{\bar n}}\right| = \left|\frac{\partial^2 U}{\partial x_{\bar n}\partial x_{\bar n}}\right| |\tau_{\bar n\bar n}|^2.$$

Therefore:

Theorem.—The Hessian of a function of n independent variables is a covariant of weight 2.

(2) Generalized Hessian.—Let

$$d^m U = \left(\frac{\partial^m U}{\partial x_{\bar n}\cdots\partial x_{\bar n}}\right)(dx_{\bar n}),$$

and

$$dx_{\bar n} = (\tau_{\bar n\bar n})(dy_{\bar n}),$$

so that

$$d^m U = \left(\frac{\partial^m U}{\partial x_{\bar n}\cdots\partial x_{\bar n}}\right) : (\tau_{\bar n\bar n})^m (dy_{\bar n})^m$$

$$= \left(\frac{\partial^m U}{\partial y_{\bar n}\cdots\partial y_{\bar n}}\right) : (dy_{\bar n})^m,$$

and consequently:

$$\left|\frac{\partial^m U}{\partial y_{\bar n}\cdots\partial y_{\bar n}}\right| = \left|\frac{\partial^m U}{\partial x_{\bar n}\cdots\partial x_{\bar n}}\right| |\tau_{\bar n\bar n}|^m.$$

Whence:

Theorem.—General Hessian $\left|\frac{\partial^m U}{\partial x_{\bar n}\cdots\partial x_{\bar n}}\right|$ of a function U of n independent variables, is a covariant of weight m.

(3) Jacobian.—Let

$$(df_{\bar n}) = \left(\frac{\partial f_{\bar n}}{\partial x_{\bar n}}\right)_{\swarrow}(dx_{\bar n}),$$

where the arrow signifies order in which subscripts are to be taken, and

$$(dx_{\bar n}) = (\tau_{\bar n\bar n})(dy_{\bar n}),$$

so that

$$(df_{\bar n}) = \left(\frac{\partial f_{\bar n}}{\partial x_{\bar n}}\right)_{\swarrow}(\tau_{\bar n\bar n})(dy_{\bar n}) = \left(\frac{\partial f_{\bar n}}{\partial y_{\bar n}}\right)_{\swarrow}(dy_{\bar n}),$$

and consequently

$$\left|\frac{\partial f_{\bar n}}{\partial y_{\bar n}}\right| = \left|\frac{\partial f_{\bar n}}{\partial x_{\bar n}}\right| \cdot |\tau_{\bar n\bar n}|.$$

Whence:

Theorem.—Jacobian $\left|\frac{\partial x_{\bar n}}{\partial f_{\bar n}}\right|$ of n independent functions of n independent variables, is a covariant of weight 1.

(4) Generalized Jacobian.—Type I.

Let

$$(d^{m-1}f_{\bar{n}}) = \left(\frac{\partial^{m-1}f_{\bar{n}}}{\partial x_{\bar{n}} \cdots \partial x_{\bar{n}}}\right)\!\!\downarrow : (dx_{\bar{n}}),$$

$$= \left(\frac{\partial^{m-1}f}{\partial x_{\bar{n}} \cdots \partial x_{\bar{n}}}\right)\!\!\downarrow : (\tau_{\bar{n}\bar{n}}\between dy_{\bar{n}})^{m-2},$$

$$= \left(\frac{\partial^{m-1}f_{\bar{n}}}{\partial y_{\bar{n}} \cdots \partial y_{\bar{n}}}\right)\!\!\downarrow : (dy)^{m-1}.$$

Accordingly

$$\left|\frac{\partial^{m-1}f_{\bar{n}}}{\partial y_{\bar{n}} \cdots \partial y_{\bar{n}}}\right| = \left|\tau_{\bar{n}\bar{n}}\right|^{m-1}\left|\frac{\partial^{m-1}f_{\bar{n}}}{\partial x_{\bar{n}} \cdots \partial x_{\bar{n}}}\right|,$$

and consequently:

Theorem.—Generalized Jacobian $\left|\dfrac{\partial^{m-1}f_{\bar{n}}}{\partial x_{\bar{n}} \cdots \partial x_{\bar{n}}}\right|$ is a covariant of weight $m - 1$.

(5) Type II. In a similar manner it may readily be shown that the generalized Jacobian

$$\left|\frac{\partial^{m-q}f_{\bar{n}}}{\partial x_{\bar{n}} \cdots \partial x_{\bar{n}}}\right|$$

is a covariant of weight $m - q$.

Note.—For a different treatment and for extensions along certain lines, the reader may consult E. R. Hedrick, *Annals of Mathematics*, Vol. I, p. 49. The above results were obtained independently, and I am indebted to Professor E. B Wilson for this reference.

The significance of the above results in connection with the theory which we are now investigating, that of the general vector function, lies in the fact that any vector field, or rather the corresponding vector function of the position vector $\boldsymbol{r}$, may be expressed as a series of homogeneous vector functions of $\boldsymbol{r}$, whose determinants are of the form of these differential covariants and invariants, being consequently independent of the axes of reference. We shall have occasion to treat this point at length at a later stage.

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